Section 14.5

Directional Derivatives and Gradients

Prelecture Review Video

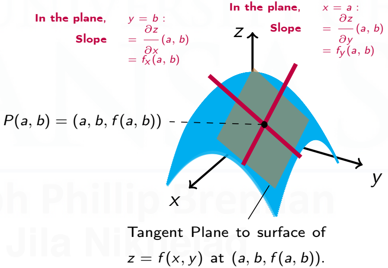
1 Directional Derivative

Directional Derivatives

Let $z = f(x, y)$, and let (a, b) be a point in the domain of f.

$f_x(a, b)$ is the rate of change in the x-direction ($\vec{\text{i}}$-direction) at (a, b).

$f_y(a, b)$ is the rate of change in the y-direction ($\vec{\text{j}}$-direction) at (a, b).



Question: What is the rate of change of $f(x, y)$ in **any** given direction?

Notation and Definition of Directional Derivatives

To answer the question, consider the line through $(a, b, f(a, b))$ with unit direction vector $\vec{u} = \langle u_1, u_2 \rangle$. This line is parametrized by

$$x(t) = a + u_1 t \qquad\qquad y(t) = b + u_2 t$$

The rate of change of f in the $\vec{u}$-direction is

$$\underbrace{D_{\vec{u}} f(a, b)}_{\text{Notation}} = \underbrace{\lim_{t \to 0} \frac{f(\vec{r}_P + t\vec{u}) - f(P)}{t}}_{\text{definition}} = \lim_{t \to 0} \frac{f(a + u_1 t, b + u_2 t) - f(a, b)}{t}$$

which is called the **directional derivative** of f in the direction $\vec{u}$.

- For example, $D_{\vec{i}} f(a, b) = f_x(a, b)$ and $D_{\vec{j}} f(a, b) = f_y(a, b)$.

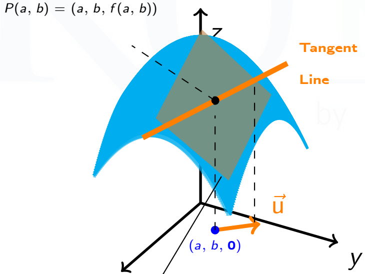
Link

Geometric Definition of Directional Derivatives

The directional derivative $D_{\vec{u}}f(a,b)$ is a scalar that measures the instantaneous rate of change, namely

$$\frac{\text{change in the value of } f(x,y)}{\text{horizontal distance traveled in direction } \vec{u}}$$

Remember that $\vec{u}$ must be a unit vector!



Tangent Plane to the Graph of $z = f(x,y)$ at $(a,b,f(a,b))$.

Since the tangent line in direction $\vec{u}$ at $(a,b,f(a,b))$ is on the tangent plane, the slope measures to be

$$\frac{\overbrace{L_{(a,b)}(a+u_1, b+u_2) - f(a,b)}^{\text{Change in } z}}{\|\vec{u}\|}$$

$$= f_x(a,b)u_1 + f_y(a,b)u_2$$

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Other Representations, Gradient Vector and Formula for the Directional Derivative

Let $\vec{u} = \langle p, q \rangle$ be a unit vector and $f(x, y)$ a differentiable function of two variables. Let $g(t) = f(a + u_1 t, b + u_2 t)$,

$$\begin{aligned} D_{\vec{u}} f(a,b) &= \lim_{t\to 0} \frac{f(a+u_1 t, b+u_2 t) - f(a,b)}{t} = \lim_{t\to 0} \frac{g(t)-g(0)}{t} \\ &= g'(0) = u_1 f_x(a,b) + u_2 f_y(a,b) \qquad \text{(Revisit after Section 14.6)} \\ &= \langle f_x(a,b),\ f_y(a,b)\rangle \cdot \vec{u}. \end{aligned}$$

The **gradient vector** of f at (a, b) is

$$\nabla f(a,b) = \langle f_x(a,b),\ f_y(a,b)\rangle$$

In terms of this new notation,

$$D_{\vec{u}} f(a,b) = \nabla f(a,b) \cdot \vec{u}$$

Directional Derivatives: Examples

Example 1: Calculate the directional derivative of

$$f(x,y) = x^3y^2 + 7x^2y^3$$

at (1,1) in the direction of the vector $\vec{v} = \langle -2, 1 \rangle$.

<u>Solution:</u> *First*, find a unit vector $\vec{u}$ parallel to $\vec{v}$:

$$\|\vec{v}\| = \sqrt{5} \qquad\qquad \vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \left\langle -\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle$$

Second, calculate the gradient vector at $(1,1)$:

$$\nabla f(x,y) = \left\langle 3x^2y^2 + 14xy^3,\ 2x^3y + 21x^2y^2 \right\rangle \qquad \nabla f(1,1) = \langle 17, 23 \rangle$$

Third, calculate the directional derivative:

$$D_{\vec{u}} f(1,1) = \langle 17, 23 \rangle \cdot \left\langle -\frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}} \right\rangle = \boxed{-\frac{11}{\sqrt{5}}}$$

Gradients and Directional Derivatives in 3 Variables

Let $f(x, y, z)$ be a function of 3 variables. The **gradient vector** of differentiable function f at a point (a, b, c) in the domain of f is

$$\nabla f(a, b, c) = \langle f_x(a, b, c),\ f_y(a, b, c),\ f_z(a, b, c)\rangle\ .$$

If $\vec{u} = \langle u_1, u_2, u_3\rangle$ is a unit vector, then the **directional derivative** of f at (a, b, c) in the direction of $\vec{u}$ is

$$D_{\vec{u}} f(a, b, c) = \lim_{t \to 0} \frac{f(a + u_1 t,\ b + u_2 t,\ c + u_3 t) - f(a, b, c)}{t}$$

$$= \nabla f(a, b, c) \cdot \vec{u}\ .$$

The definitions are similar for functions in any number of variables.

▸ Directional Derivative in Picture

Directional Derivatives: Examples

Example 2: Calculate the directional derivative of

$$f(x, y, z) = x^3 - xy^2 - z$$

at (1,1,0) in the direction of the vector $\vec{v} = \langle 2, -3, 6 \rangle$.

<u>Solution:</u> *First*, normalize $\vec{v}$ to a unit vector $\vec{u}$:

$$\|\vec{v}\| = 7 \qquad\qquad \vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \left\langle \frac{2}{7},\ \frac{-3}{7},\ \frac{6}{7} \right\rangle$$

Second, calculate the gradient vector at $(1, 1, 0)$:

$$\nabla f(x, y, z) = \left\langle 3x^2 - y^2,\ -2xy,\ -1 \right\rangle \qquad\qquad \nabla f(1, 1, 0) = \langle 2, -2, -1 \rangle$$

Third, calculate the directional derivative:

$$D_{\vec{u}} f(1, 1, 0) = \langle 2, -2, -1 \rangle \cdot \left\langle \frac{2}{7},\ \frac{-3}{7},\ \frac{6}{7} \right\rangle = \boxed{\frac{4}{7}}$$

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2 Directions of Fastest Increase and Fastest Decrease

Directions with Extreme Rates of Change

Suppose that f is a differentiable function of two variables and $\vec{u}$ is a unit vector.

$$D_{\vec{u}} f(a, b) = \nabla f(a, b) \cdot \vec{u} = \|\nabla f(a, b)\| \, \|\vec{u}\| \cos(\theta) = \underbrace{\|\nabla f(a, b)\|}_{\text{nonnegative}} \cos(\theta)$$

where θ is the angle between $\nabla f(a, b)$ and $\vec{u}$. **Therefore...**

The gradient ∇f points in the direction that f is increasing fastest.

- I.e., the **largest** (**smallest**) directional derivative is in the direction $\nabla f(a, b)$ (or $-\nabla f(a, b)$) and equal to $\|\nabla f(a, b)\|$ (or $-\|\nabla f(a, b)\|$).

  (This assumes $\nabla f(a, b) \neq \vec{0}$. What if $\nabla f(a, b) = \vec{0}$? Stay tuned!)

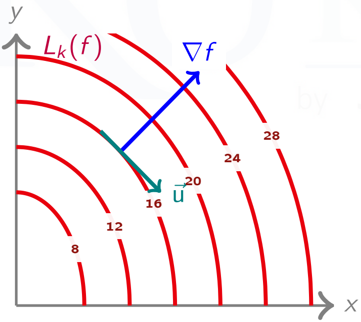
- The directional derivative is **zero** in any direction **orthogonal** to $\nabla f(a, b)$.

Gradients and Level Sets

Remember that the level curves of $f(x, y)$ are the curves where f is constant. If $f(a, b) = k$, then (a, b) is on the level curve

$$L_k(f) = \{(x, y) \mid f(x, y) = k\}$$

$\nabla f(a, b)$ is orthogonal to the tangent line of $L_k(f)$ at (a, b).



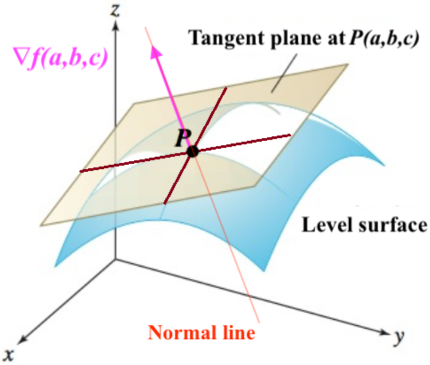
Reason: Moving along the level curve does not change the value of f. So $D_{\vec{u}} f(P) = 0$, where $\vec{u}$ points along the tangent line to the level curve. That is, $\nabla f(a, b) \perp \vec{u}$.

Visual Interpretation: Moving perpendicularly to the tangent line is the fastest way to move between level curves.

Gradients and Level Sets

The situation is similar for a function $f(x, y, z)$ of three variables.

- $\nabla f(a, b, c)$ = direction of greatest increase of f at (a, b, c)
- $D_{\vec{u}} f(a, b, c) = 0$ in any direction $\vec{u}$ tangent to the level surface $L_k(f)$ (that is, if $\vec{u}$ lies in the tangent plane of $L_k(f)$)
- $\nabla f(a, b, c)$ is orthogonal to the tangent plane
- **Normal line** to $L_k(f)$: the line through (a, b, c) with direction vector $\nabla f(a, b, c)$.



Directions with Extreme Rates of Change

Example 3: A metal surface S is shaped like the graph of $z = 2x^2 - xy + 4y^2 - 3y$. A marble is placed on the surface at the point $P(1, 1, 2)$.

Part 1: Which way does the marble start to roll?

Solution: We want to find the direction of fastest *decrease* of f.

$$\nabla f(x, y) = \langle 4x - y,\ -x + 8y - 3 \rangle \qquad \nabla f(1, 1) = \langle 3, 4 \rangle$$

So the direction is $\langle -3, -4 \rangle$ (or $\left\langle -\frac{3}{5}, -\frac{4}{5} \right\rangle$ if you want a unit vector).

Part 2: Find a horizontal tangent line to S at P.

Solution: The direction vector is orthogonal to the gradient; use $\langle 4, -3 \rangle$. So the line can be written as

$$\vec{r}(t) = \langle 1 + 4t,\ 1 - 3t,\ 2 \rangle$$

▸ Video

3 Tangent Planes and Normal Lines

Tangent Planes and Normal Lines

- We can find the tangent plane to any surface S defined by an equation in x, y, z (we do not need z to be a function of x and y).
- Express the equation in the form $\underbrace{F(x, y, z) = \text{constant}}_{\text{A level Surface of } F}$
- Next, compute $\nabla F(x, y, z)$.
- Then, the equation of the tangent plane at any point (a, b, c) on S is

$$\nabla F(a, b, c) \cdot \langle x - a,\ y - b,\ z - c \rangle = 0$$

and the normal line has equation

$$\vec{r}(t) = \langle a, b, c \rangle + t\nabla F(a, b, c)$$

or equivalently

$$x = a + tF_x(a, b, c), \quad y = b + tF_y(a, b, c), \quad z = c + tF_z(a, b, c).$$

Tangent Planes and Normal Lines

Example 4: Find the tangent plane and the normal line to the surface $4x^2 + 9y^2 - z^2 = 16$ at $(2, 1, 3)$.

Solution: Let $F(x, y, z) = 4x^2 + 9y^2 - z^2$, so the surface is $F(x, y, z) = 16$.

$$\nabla F(x, y, z) = \langle 8x,\ 18y,\ -2z\rangle \qquad \nabla F(2, 1, 3) = \langle 16,\ 18,\ -6\rangle$$

Tangent Plane:

$$16(x - 2) + 18(y - 1) - 6(z - 3) = 0$$

Normal Line:

$$\vec{r}(t) = \langle 2, 1, 3\rangle + t\,\langle 16, 18, -6\rangle$$